7.2 Finding Volume using the Washer Method

Example 1) Find the volume of the solid formed by revolving the region bounded by the graphs $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

\[
V = \pi \int_0^1 \left[R^2 - r^2\right] \, dx
\]

\[
= \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] \, dx
\]

\[
= \pi \int_0^1 (x - x^4) \, dx
\]

\[
= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1
\]

\[
= \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}
\]
This application of the method of slicing is called the **washer method**. The shape of the slice is a circle with a hole in it, so we subtract the area of the inner circle from the area of the outer circle.

**Washer Method Formula**

\[ V = \pi \int_{a}^{b} \left( R^2 - r^2 \right) dx \]

**OR**

\[ V = \int_{a}^{b} \left( \pi R^2 - \pi r^2 \right) dx \]
Example 2) Find the volume of the solid enclosed by the curves $y = x$ and $y = x^2$ when it is rotated about the y-axis.

$$V = \pi \int_a^b R^2 - r^2 \, dy$$

$$V = \pi \int_0^1 [(\sqrt{y})^2 - (y)^2] \, dy$$

$$= \frac{\pi}{6}$$

What if the line was rotated about the line $x = -1$?

$$R = \sqrt{y} + 1$$

$$r = y + 1$$

$$V = \pi \int_a^b [(\sqrt{y} + 1)^2 - (y + 1)^2] \, dy$$

$$\frac{\pi}{2}$$
Example 3) Find the volume of the region bounded by the curves $y = x$ and $y = x^2$ when it is rotated about the line $y = 3$.

$$R = 3 - x^2$$

$$r = 3 - x$$

$$V = \pi \int_0^1 [(3-x^2)^2 - (3-x)^2] \, dx$$

$$= \frac{13\pi}{15}$$
**Example 4)** The region bounded by \( y = x^2 \) and \( y = 2x \) is revolved about the y-axis. Find the volume.

If we use a horizontal slice, the disk now has a hole in it, making it a washer.

The volume of the washer is:

\[
\left( \pi R^2 - \pi r^2 \right) \cdot \text{thickness}
\]

\[
= \pi \left( R^2 - r^2 \right) \, dy
\]

\[
V = \int_0^4 \pi \left[ (\sqrt{y})^2 - \left( \frac{y}{2} \right)^2 \right] \, dy
\]
**Example 5)** The region to the right is enclosed by \( x = 0, \ y = 0, \ x = 1 \) and \( y = x^2 + 1 \).

What is the volume of the solid formed by revolving this region about the \( x \)-axis?

What about if we revolved it around the \( y \)-axis?
Example 7) Find the volume of the region bounded between $x = 0$, $y = 1$, $x = 1$, $y = (x-1)^2 + 1$ when it is rotated....

a) about the x-axis

$$R = (x-1)^2 + 1$$

$$r = 1$$

b) about $x = -1$

$$R = (1+\sqrt{y-1}) + 1$$

$$r = 1$$

c) about $x = 2$

$$R = 2$$

$$r = 2 - (1+\sqrt{y-1})$$